Student Notes

Student Study Session Topic: Writing Justifications and explanations on the AP Calculus Exams

The goal of this prep session is to help you develop a method for approaching those AP Calculus questions include the instructions to

- Justify your answer
- Explain your reasoning
- Explain the meaning of a definite integral

Here's the way to approach the question:

1. **Identify the theorem(s) or definition(s) that might be used**. To identify the theorem you need to know the hypotheses and conclusions. The wording of the question may mimic the *conclusion* of the appropriate theorem.

   The theorems that are most often used are these:
   - First derivative test
   - Second derivative test
   - Candidates' test (a combination of the first derivative test and the extreme value theorem)
   - Extreme value theorem
   - Intermediate value theorem
   - Mean value theorem

   The statement of theses theorems and some hints on learning theorems in general are at the end of this handout.

2. Having identified the theorem that applies, **make sure that the hypotheses are met**. This may require some computations that you should show on you paper.

3. **Tie these together with a short sentenc**e.

<u>Practice #1:</u> 2008 BC 5 (a) Answer this question and including the justification.

The derivative of a function *f* is given by $f'(x) = (x-3)e^x$ for $x > 0$,…

(a) The function *f* has a critical point at $x = 3$. At this point, does *f* have a relative minimum, a relative maximum or neither? Justify your answer.

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Compare your answers to the ***wrong*** answers on the slides:

1. The function has a relative minimum at $x = 3$ because the function decreases then increases.

2. At $x = 3$ the function has a relative minimum because the slope changes from negative to positive at $x = 3$.

3. f has a relative minimum at $x = 3$ because as the first derivative test shows, the slope of f before $x = 3$ is decreasing and after $x = 3$, the slope of f is increasing. Therefore, there is a relative minimum at $x = 3$.

4. The student calculated the slope at $x = 2$ (negative) and at $x = 4$ (positive) and wrote: Since the value goes from negative to positive $x = 3$ is a relative minimum.

5. A relative minimum because it goes from negative to positive.

So what does a correct answer look like?

For all $0 < x < 3$, the derivative is negative so the function is decreasing here and for all $x > 3$ the derivative is positive so the function is increasing here. Therefore $x = 3$ is a relative minimum.

The student calculated $f''(3) > 0$ and wrote: By the second derivative test *x* = 3 is a minimum. When the second derivative is positive *f* is concave up, so the extrema must be a minimum.

The derivative of *f* changes from negative to positive at *x* = 3, so this is a minimum.

The scoring standard reads:

$f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$ Therefore, *f* has a relative minimum at $x = 3$

The question of justifying an extreme value is asked so often that a template might be in order (change the underlined parts as needed):

The function *f* has an absolute <u>maximum</u> at *x* = *a*, because f' changes from <u>positive to negative</u> only at *x* = *a*.

Or

$\underline{f'(x) > 0}$ for $a < x < b$ and $\underline{f'(x) < 0}$ for $b < x < c$. Therefore, *f* has a relative <u>maximum</u> at *x* = *b*.

Or

The derivative of *f* changes from <u>positive to negative</u> at *x* = *b*, so this is a <u>maximum</u>.

Absolute Extreme Values

For an *absolute* extreme value you must give an argument that includes the *entire domain* of the function including the endpoints. The fact that the derivative changes sign is not enough; it may change sign elsewhere and therefore have other extreme values. Also by the *Extreme Value Theorem* the endpoints are candidates for absolute extreme value and must be checked.

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The *Candidates' Test*: Another way to justify an absolute extreme value on a *closed* interval is to actually find the value at *each critical point* and at the *end points of the domain.* Then simply pick the absolute minimum or maximum. A listing of the values at all the required points is sufficient.

Look at the scoring standards for **2005 AB 2 (c)** to see how the candidates' test is used.

Practice #2: Return to **2008 BC 5** and write justifications for **(b)**: On what intervals, if any, is the graph of *f* both decreasing and concave up? Explain your reasoning.

Justifying a "No" or that a conclusion is false:

Practice #3: Write a justification for **2005 AB5/BC5 (b)** and **(d)**.

A conclusion is false because one or more of the hypotheses of a definition or theorem is not met. Saying that is not quite enough: say which hypothesis is not true. See for example

- In (b) $v'(4)$ does not exist because the definition of derivative requires that the two one-sided limits be equal. Showing they are not equal is enough to explain why $v'(4)$ does not exist.
- In (d) the Mean Value Theorem does not apply because one of its hypotheses requires that the function be differentiable on the interior of the interval; our function is not.
- A *counterexample* may also be used. A counterexample is an example that meets all the hypotheses given in the question, but for which the alleged conclusion is not true. This disproves the statement.

Recognizing the theorem to use is the first step.

<u>**Practice # 4:**</u> Now try **2007 AB 3** (a) and (b). Don't skip this easy question. The mean on this question was 0.97 out of 9 points; the median was 0.

The meaning of a definite integral

<u>**Practice # 5:**</u> Another common question is to **explain the meaning of a definite integral** in the context of the problem. A full answer explains (1) what the answer gives, (2) the correct units and (3) includes the meaning of the limits of integration.

- Explain the 2 integrals in **2003 AB 3 (d)**

Attached to this handout are these questions:

2008 BC5 (extreme value),

2005 AB 2 (c) (Candidates' test),

2005 AB5/BC5 (MVT)

2007 AB 3 (IVT, MVT),

2003 AB3 (definite integrals)

Appendix 1: The main theorems tested on the AP Calculus exams are listed. There may be others.

Intermediate value theorem:

If f is continuous on a closed interval [a, b] and $f\left(a\right) \neq f\left(b\right)$, then for every (any, all) number M between $f\left(a\right)$ and $f\left(b\right)$, there exist at least one number c in the open interval (a, b), such that $f\left(c\right) = M$

In other words, a continuous function, f takes on all the values between $f\left(a\right)$ and $f\left(b\right)$.

Extreme value theorem

If f is continuous on a closed interval [a, b], then f has a maximum and a minimum value on that interval.

A more formal wording asserts that there is a value in the domain for which the function assumes a value larger (1) or smaller (2) than all the other values:

If f is continuous on the closed interval [a, b], then:

1. there exists a number c_1 in [a, b] such that $f\left(x\right) \le f\left(c_1\right)$ for all x in [a, b]
2. there exists a number c_2 in [a, b] such that $f\left(x\right) \ge f\left(c_2\right)$ for all x in [a, b]

Note (1) that a function may attain its maximum and minimum value more than once. For example, the maximum value of $y = \sin\left(x\right)$ is 1 and it reaches this value many, many times; and (2) the extreme values often occur at the endpoint of the domain.

Mean value theorem and Rolle's theorem

Rolle: If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b) and $f\left(a\right) = f\left(b\right)$, then there exists a number c in the open interval (a, b) such that $f'\left(c\right) = 0$

MVT: If f is continuous on the closed interval [a, b], and differentiable on the open interval (a, b), then there exists a number c in the open interval (a, b) such that $f'\left(c\right) = \dfrac{f\left(b\right) - f\left(a\right)}{b - a}$.

Graphically the MVT means that somewhere between the endpoints of a differentiable function there is a tangent line parallel to the segment between the endpoints.

Student Notes

Increasing/Decreasing theorem

If f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), and if for all c in the open interval (a, b), $f'(c) > 0$ ($f'(c) < 0$), then f is increasing (decreasing) on the closed interval $[a, b]$.

Note: If $f'(c) = 0$, this theorem does not apply. Why?

First derivative test

If f is differentiable and c is a critical point of f and if $f'(x)$ changes from *positive to negative* at $x = c$, then $f(c)$ is a local maximum of f.

If f is differentiable and c is a critical point of f and if $f'(x)$ changes from *negative to positive* at $x = c$, then $f(c)$ is a local minimum of f.

Second derivative test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c, then if $f''(c) > 0$, f has a local *minimum* value at $x = c$.

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c, then if $f''(c) < 0$, f has a local *maximum* value at $x = c$.

If $f''(c) = 0$, the second derivative test cannot be used. The function may have a maximum, a minimum or neither at c.

Think: concavity. Concave up, minimum; concave down, maximum.

Appendix 2: Learning Theorems

These theorems should be understood more than memorized. Understand their structure and learn to "play" with them. Learn their meaning by looking at them *graphically*, *numerically*, *analytically* and *verbally*.

To "play" with a theorem: The assumptions matter. Each hypothesis is necessary for some reason; understand why each is there. Change one of the hypotheses and see what happens and what changes when the hypothesis changes. What would be different if the hypothesis were not true.

Check the *converse*, *inverse*, and *contrapositive*.

Learn about *counterexamples*.

The words “any”, “every” and “all” are used in many theorems and definitions. The three words are always interchangeable: read the theorem and replace one of these words with each of the others. This may help you better understand what the theorem or definition means.

- For *all* real numbers x, $x^2 \geq 0$
- For *any* real number x, $x^2 \geq 0$
- For *every* real number x, $x^2 \geq 0$

Proofs are, believe it or not, less important to understanding the theorem. If you’re convinced a theorem is true, or can see why it’s true from the diagram, then you can scan or even skip the proof.

Understanding a theorem is different than understanding its proof. A reason to carefully study proofs is to learn how to do proofs yourself. Some proofs help you understand the theorem better (some do not).

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$ Therefore, f has a relative minimum at $x = 3$.	$2: \begin{cases} 1: \text{minimum at } x = 3 \\ 1: \text{justification} \end{cases}$
(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$ $f''(x) > 0$ for $x > 2$ $f'(x) < 0$ for $0 < x < 3$ Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.	$3: \begin{cases} 2: f''(x) \\ 1: \text{answer with reason} \end{cases}$
(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$ $u = x - 3 \quad dv = e^x\,dx$ $du = dx \quad v = e^x$ $f(3) = 7 + (x-3)e^x\Big\vert_1^3 - \int_1^3 e^x\,dx$ $= 7 + \left((x-3)e^x - e^x\right)\Big\vert_1^3$ $= 7 + 3e - e^3$	$4: \begin{cases} 1: \text{uses initial condition} \\ 2: \text{integration by parts} \\ 1: \text{answer} \end{cases}$

Question 2

The tide removes sand from Sandy Point Beach at a rate modeled by the function R, given by

$$R(t) = 2 + 5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t) = \frac{15t}{1 + 3t}.$$

Both $R(t)$ and $S(t)$ have units of cubic yards per hour and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

(a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

(b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

(c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

(d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

(a) $\int_0^6 R(t)\,dt = 31.815$ or 31.816 yd^3

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer with units} \end{cases}$

(b) $Y(t) = 2500 + \int_0^t (S(x) - R(x))\,dx$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(c) $Y'(t) = S(t) - R(t)$

$Y'(4) = S(4) - R(4) = -1.908$ or -1.909 yd^3/hr

1 : answer

(d) $Y'(t) = 0$ when $S(t) - R(t) = 0$.
The only value in $[0, 6]$ to satisfy $S(t) = R(t)$ is $a = 5.117865$.

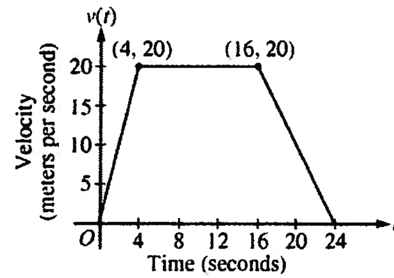
t	$Y(t)$
0	2500
a	2492.3694
6	2493.2766

The amount of sand is a minimum when $t = 5.117$ or 5.118 hours. The minimum value is 2492.369 cubic yards.

$3 : \begin{cases} 1 : \text{sets } Y'(t) = 0 \\ 1 : \text{critical } t\text{-value} \\ 1 : \text{answer with justification} \end{cases}$

Question 5

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.



(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$

The car travels 360 meters in these 24 seconds.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{meaning with units} \end{cases}$

(b) $v'(4)$ does not exist because

$$\lim_{t \to 4^-} \left(\frac{v(t) - v(4)}{t - 4} \right) = 5 \ne 0 = \lim_{t \to 4^+} \left(\frac{v(t) - v(4)}{t - 4} \right).$$

$$v'(20) = \frac{20 - 0}{16 - 24} = -\frac{5}{2} \text{ m/sec}^2$$

$3 : \begin{cases} 1 : v'(4) \text{ does not exist, with explanation} \\ 1 : v'(20) \\ 1 : \text{units} \end{cases}$

(c)

$$a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$2 : \begin{cases} 1 : \text{finds the values } 5, 0, -\frac{5}{2} \\ 1 : \text{identifies constants with correct intervals} \end{cases}$

(d) The average rate of change of v on $[8, 20]$ is

$$\frac{v(20) - v(8)}{20 - 8} = -\frac{5}{6} \text{ m/sec}^2.$$

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

$2 : \begin{cases} 1 : \text{average rate of change of } v \text{ on } [8, 20] \\ 1 : \text{answer with explanation} \end{cases}$

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	4	2	5
2	9	2	3	1
3	10	-4	4	2
4	-1	3	6	7

The functions f and g are differentiable for all real numbers, and g is strictly increasing. The table above gives values of the functions and their first derivatives at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

(b) Explain why there must be a value c for $1 < c < 3$ such that $h'(c) = -5$.

(c) Let w be the function given by $w(x) = \int_1^{g(x)} f(t)\,dt$. Find the value of $w'(3)$.

(d) If g^{-1} is the inverse function of g, write an equation for the line tangent to the graph of $y = g^{-1}(x)$ at $x = 2$.

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.

$$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$$

(b) $\dfrac{h(3) - h(1)}{3 - 1} = \dfrac{-7 - 3}{3 - 1} = -5$
Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.

$$2 : \begin{cases} 1 : \dfrac{h(3) - h(1)}{3 - 1} \\ 1 : \text{conclusion, using MVT} \end{cases}$$

(c) $w'(3) = f(g(3)) \cdot g'(3) = f(4) \cdot 2 = -2$

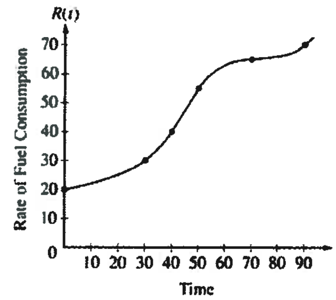
$$2 : \begin{cases} 1 : \text{apply chain rule} \\ 1 : \text{answer} \end{cases}$$

(d) $g(1) = 2$, so $g^{-1}(2) = 1$.
$$\left(g^{-1}\right)'(2) = \frac{1}{g'\left(g^{-1}(2)\right)} = \frac{1}{g'(1)} = \frac{1}{5}$$
An equation of the tangent line is $y - 1 = \frac{1}{5}(x - 2)$.

$$3 : \begin{cases} 1 : g^{-1}(2) \\ 1 : \left(g^{-1}\right)'(2) \\ 1 : \text{tangent line equation} \end{cases}$$

Question 3

The rate of fuel consumption, in gallons per minute, recorded during an airplane flight is given by a twice-differentiable and strictly increasing function R of time t. The graph of R and a table of selected values of $R(t)$, for the time interval $0 \le t \le 90$ minutes, are shown above.



t (minutes)	$R(t)$ (gallons per minute)
0	20
30	30
40	40
50	55
70	65
90	70

(a) Use data from the table to find an approximation for $R'(45)$. Show the computations that lead to your answer. Indicate units of measure.

(b) The rate of fuel consumption is increasing fastest at time $t = 45$ minutes. What is the value of $R''(45)$? Explain your reasoning.

(c) Approximate the value of $\int_0^{90} R(t)\,dt$ using a left Riemann sum with the five subintervals indicated by the data in the table. Is this numerical approximation less than the value of $\int_0^{90} R(t)\,dt$? Explain your reasoning.

(d) For $0 < b \le 90$ minutes, explain the meaning of $\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Explain the meaning of $\frac{1}{b}\int_0^b R(t)\,dt$ in terms of fuel consumption for the plane. Indicate units of measure in both answers.

(a) $R'(45) \approx \dfrac{R(50) - R(40)}{50 - 40} = \dfrac{55 - 40}{10}$
$= 1.5 \text{ gal/min}^2$

2 :
- 1 : a difference quotient using numbers from table and interval that contains 45
- 1 : 1.5 gal/min^2

(b) $R''(45) = 0$ since $R'(t)$ has a maximum at $t = 45$.

2 :
- 1 : $R''(45) = 0$
- 1 : reason

(c) $\int_0^{90} R(t)\,dt \approx (30)(20) + (10)(30) + (10)(40)$
$+(20)(55) + (20)(65) = 3700$

Yes, this approximation is less because the graph of R is increasing on the interval.

2 :
- 1 : value of left Riemann sum
- 1 : "less" with reason

(d) $\int_0^b R(t)\,dt$ is the total amount of fuel in gallons consumed for the first b minutes.

$\frac{1}{b}\int_0^b R(t)\,dt$ is the average value of the rate of fuel consumption in gallons/min during the first b minutes.

3 :
- 2 : meanings
  - 1 : meaning of $\int_0^b R(t)\,dt$
  - 1 : meaning of $\frac{1}{b}\int_0^b R(t)\,dt$
  - $<-1>$ if no reference to time b
- 1 : units in both answers